§ $I\times S^2$

[An Introduction to Riemannian Geometry **Jose Natario**] p.139 Ex2.8 (6)

(6) Consider the metric

$$g = A^2(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta \, d\varphi \otimes d\varphi$$

on $M = I \times S^2$, where r is a local coordinate on $I \subset \mathbb{R}$ and (θ, φ) are spherical local coordinates on S^2.

(a) Compute the Ricci tensor and the scalar curvature of this metric.
(b) What happens when $A(r) = (1-r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to S^3)?

(c) And when $A(r) = (1+r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to the **hyperbolic** 3-**space**)?
(d) For which functions $A(r)$ is the scalar curvature constant?

(M, g)是一個黎曼流形

$$X_1 = \frac{\partial}{\partial r}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \varphi}$$

$< X_1, X_1 >= A(r)^2, < X_2, X_2 >= r^2, < X_3, X_3 >= r^2 \sin^2\theta$ ，所以取 orthonormal frames

為 $E_1 = \frac{1}{A}\frac{\partial}{\partial r}, E_2 = \frac{1}{r}\frac{\partial}{\partial \theta}, E_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$

An orthonormal coframes $\omega^1 = A(r)dr, \omega^2 = rd\theta, \omega^3 = r\sin\theta d\varphi$

由結構方程式 $d\omega^i = \omega^j \wedge \omega_j^i$

$$d\omega^1 = 0$$

$$d\omega^2 = dr \wedge d\theta = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 = Adr \wedge \omega_1^2$$

$$\therefore \omega_1^2 = \frac{1}{A}d\theta$$

$$d\omega^3 = \sin\theta dr \wedge d\theta + r\cos\theta d\theta \wedge d\varphi = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3$$

$$= Adr \wedge \omega_1^3 + rd\theta \wedge \omega_2^3 \text{ ， } \therefore \omega_1^3 = \frac{\sin\theta}{A}d\varphi, \omega_2^3 = \cos\theta d\varphi$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = d(\frac{1}{A}d\theta) - (\frac{\sin\theta}{A}d\varphi) \wedge (-\cos\theta d\varphi) = \frac{-A'}{A^2}dr \wedge d\theta = \frac{-A'}{rA^3}\omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^2 - \omega_1^2 \wedge \omega_2^3 = d(\frac{\sin\theta}{A} d\varphi) - (\frac{1}{A} d\theta) \wedge (\cos\theta d\varphi)$$

$$= (\frac{-A'\sin\theta}{A^2} dr + \frac{\cos\theta}{A} d\theta) \wedge d\varphi - \frac{\cos\theta}{A} d\theta \wedge d\varphi$$

$$= \frac{-A'\sin\theta}{A^2} d\theta \wedge d\varphi = \frac{-A'}{rA^3} \omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = d(\cos\theta d\varphi) - (-\frac{1}{A} d\theta) \wedge (\frac{\sin\theta}{A} d\varphi)$$

$$= (-\sin\theta d\theta \wedge d\varphi) + \frac{\sin\theta}{A^2} d\theta \wedge d\varphi$$

$$= (\frac{\sin\theta}{A^2} - \sin\theta) d\theta \wedge d\varphi = \frac{1}{r^2} (\frac{1}{A^2} - 1) \omega^2 \wedge \omega^3$$

$$R_{iji}^j = \Omega_i^j (E_i, E_j) \text{ ， } R_{ij} = R_{kij}^k$$

$$R_{121}^2 = \Omega_1^2 (E_1, E_2) = \frac{-A'}{rA^3} \text{ ， } R_{131}^3 = \Omega_1^3 (E_1, E_3) = \frac{-A'}{rA^3} \text{ ， } R_{232}^3 = \Omega_2^3 (E_2, E_3) = \frac{1}{r^2} (\frac{1}{A} - 1)$$

$$R_{11} = R_{111}^1 + R_{211}^2 + R_{311}^3 = -R_{121}^2 - R_{131}^3 = \frac{2A'}{rA^3}$$

$$R_{22} = R_{122}^1 + R_{222}^2 + R_{322}^3 = -R_{121}^2 - R_{232}^3 = \frac{A'}{rA^3} - \frac{1}{r^2} (\frac{1}{A^2} - 1)$$

$$R_{33} = R_{133}^1 + R_{233}^2 + R_{333}^3 = -R_{131}^3 - R_{232}^3 = \frac{A'}{rA^3} - \frac{1}{r^2} (\frac{1}{A^2} - 1)$$

Scalar curvature $R = R_{11} + R_{22} + R_{33} = \frac{4A'}{rA^3} - \frac{2}{r} (\frac{1}{A^2} - 1)$